Spin glasses on Bethe Lattices for large coordination number

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Abstract

We study spin glasses on random lattices with finite connectivity. In the infinite connectivity limit they reduce to the Sherrington Kirkpatrick model. In this paper we investigate the expansion around the high connectivity limit. Within the replica symmetry breaking scheme at two steps, we compute the free energy at the first order in the expansion in inverse powers of the average connectivity (z), both for the fixed connectivity and for the fluctuating connectivity random lattices. It is well known that the coefficient of the $1/z$ correction for the free energy is divergent at low temperatures if computed in the one step approximation. We find that this annoying divergence becomes much smaller if computed in the framework of the more accurate two steps breaking. Comparing the temperature dependance of the coefficients of this divergence in the replica symmetric, one step and two steps replica symmetry breaking, we conclude that this divergence is an artefact due to the use of a finite number of steps of replica symmetry breaking. The $1/z$ expansion is well defined also in the zero temperature limit.

1 Introduction

Many studies have been devoted to finding analytic solutions of more realistic models than the Sherrington-Kirkpatrick one. The diluted spin glass models belong to this class, and they are characterized by a finite coordination number; these models are also interesting because they are connected with different optimization problems [1].

In the present work we consider lattices where each site is connected with a finite number of randomly chosen sites; we study both the cases where the connectivity is fixed and where the connectivity is a Poissonian variable with given mean value. The spin interaction is only among nearest neighbour pairs.

The random structure of these lattices allows us to neglect the probability of closed paths of finite length: this probability becomes indeed zero in the thermodynamic limit: the correlations among the neighbours of a given spin can be neglected. We are therefore dealing with mean field models, although the difficulties due to the finite connectivity don't allow us to solve them exactly.

Whereas in the SK model only the overlap between two replicas occurs as order parameter (the order parameter is a function in the infinite step replica symmetry breaking solution and a pure number when the replica symmetry is exact), in the finite connectivity models the order parameter becomes a function of the overlaps of any number of replicas and then it becomes a function of an infinite number of variables when the symmetry is totally broken; as a result it is extremely difficult to find the exact free energy [2, 3, 4]. In other words the probability distribution of the effective cavity fields is Gaussian in the SK model as a consequence of the central limit theorem, so it can be characterized by its variance. When the number of neighbours z is finite, this distribution is no more Gaussian and all the moments are relevant and this leads to the presence of an infinite numbers of order parameters (also in the replica symmetric situation).

Perturbative solutions have been investigated both near the critical temperature [3] and near the infinite connectivity point (SK model) [6], [7], [5]. Recently it has been proposed proposed a general non perturbative solution developing the Bethe-Peierls cavity method to an approximation that is equivalent to a one step replica symmetry breaking level [9].

The present work addresses to the study of the large connectivity expansion: we compute the first order of the expansion in the inverse power of the connectivity (z) for the free energy. The $1/z$ expansion has been studied for the fixed connectivity model by Goldschmidt and De Dominicis, at the first step of replica symmetry breaking [7]; they found results that exhibit a low temperature divergence for the first order correction in $1/z$ to the free energy density. The $1/\sqrt{z}$ expansion, that they computed a $T = 0$ and to the second step of replica symmetry breaking, has finite corrections both at the first and the second order [5]. The $1/\sqrt{z}$ coefficient becomes yet smaller by a factor ten and by a factor three when one goes from the replica symmetric solution to the 1RSB one and from this to the 2RSB one respectively.

In this paper we have computed the coefficient of the $1/z$ expansion up to the second step of replica symmetry breaking. For the replica symmetric and the 1RSB solutions our results agree with those found by Goldschmidt and De Dominicis. These results suggest that the pathological behaviour (i.e. the low temperature divergence) is a consequence of the fact that

one stops the computation at a finite step of the iterative process for breaking the replica symmetry. Our results indicate that the $1/z$ expansion is well defined and can be used also in the zero temperature limit. We notice that a well defined $1/z$ expansion is possible for a model with continuos varying coupling only if (irrespective of the sign) the zero temperature entropy is zero in the limit $z \to \infty$. Indeed it is easy to prove using the approach of [9] that in the mean field approximation for finite z the zero temperature entropy is identically zero, so that the two limits $z \to \infty$ and $T \to 0$ could not be exchanged in an hypothetical model with continuos coupling if the zero temperature entropy were different from zero at $z = \infty$.

The paper is organized as follows: in the second section we present the two models we study and the high connectivity expansion is obtained. We use a simple method to evaluate sums over multiple replicas overlaps. In the third section we show how to perform sums over the replica indices in a simple way and in section four we illustrate our numerical results for the two steps replica symmetry breaking and we compare them with the known ones at the first step of replica symmetry breaking. Finally we present our conclusions. The appendix is devoted to a consistency check for the form of the free energy we use.

2 The large connectivity expansion.

By definition a Bethe lattice is a lattice where the Bethe-Peierls approximation is exact; this is equivalent to saying that there are no finite size loops.

In the random lattices we study the typical length of a loop is proportional to $\log N$: in the infinite volume limit it is therefore a Bethe lattice. This is locally equivalent to a tree-like structure, nevertheless by defining the Bethe lattice as a random lattice one bypasses the problem of fixing the boundary conditions to introduce frustration (this is provided by the loops of size $\sim \ln N$).

2.1 Random lattice with fixed connectivity

We therefore follow a variational formulation using in the framework of the replica approach the same scheme [7, 8]. We define a functional and we show that the free energy is obtained as the stationary point with respect to an order parameter that will be defined. The free energy functional is [7]:

$$\begin{aligned} n\beta f_n(g_n) &\equiv z \ln \left(Tr_{\{\sigma_a\}} g_n^{z+1}(\{\sigma_a\})\right) + \\ &- \frac{z+1}{2} \ln \left\{ \int_{-\infty}^{+\infty} dJ P(J) Tr_{\{\sigma_a\}} Tr_{\{\tau_a\}} g_n^z(\{\sigma_a\}) g_n^z(\{\tau_a\}) \exp\left[\beta J \sum_{a=1}^{n} \sigma_a \tau_a\right] \right\} \end{aligned} \tag{1}$$

where $Tr_{\{\sigma_a\}}$ is the sum over the 2^n configurations of the variables σ_a with $a = 1, \cdots, n$, $z+1$ is the lattice connectivity, $g_n(\{\sigma_a\})$ is a function of the n variables σ_a and it plays the role of the order parameter. Our goal is to make stationary the functional f_n with respect to the variation of $g_n(\{\sigma_a\})$. We have then to find the solution of the equation:

$$\frac{\delta f}{\delta g_n} = 0 \ , \tag{2}$$

that gives for the order parameter the equation:

$$g_n(\{\sigma_a\}) = C \int_{-\infty}^{+\infty} dJ P(J) \sum_{\{\tau_a\}} \exp\left(\sum_{a=1}^{n} \beta J \sigma_a \tau_a\right) g_n^z(\{\tau_a\}) \tag{3}$$

with

$$C = \frac{Tr_{\sigma_a} g_n^{z+1}(\{\sigma_a\})}{\int_{-\infty}^{+\infty} dJ P(J) Tr_{\{\sigma_a\}} Tr_{\{\tau_a\}} g_n^z(\{\sigma_a\}) g_n^z(\{\tau_a\}) \exp\left[\beta J \sum_{a=1}^{n} \sigma_a \tau_a\right]} \ . \tag{4}$$

We can notice that the functional defined by (1) is independent of the g_n normalization; we can then use a convenient one, provided one changes the constant C (in (4)) in $Cd^{(1-z)}$ when changing g_n in $g_n d$. The correctness of this functional (1) has been proved by De Dominicis et al. [7]; a simple way to get this result is reported for completeness in the appendix.

In order to write the order parameter $g_n(\{\sigma_a\})$ in a more explicit form (where the multiple overlaps appear) we generalize the identity

$$\exp(\beta J \sigma_a \sigma_b) = \cosh(\beta J)(1 + \sigma_a \sigma_b \tanh(\beta J)) \tag{5}$$

to

$$\exp\left(\beta J \sum_{a=1}^{n} \sigma_a \sigma_b\right) = \cosh^n(\beta J) \sum_{r=0}^{n} \left(\tanh^r(\beta J) \sum_{a_1 < \cdots < a_r} \sigma_{a_1} \tau_{a_1} \cdots \sigma_{a_r} \tau_{a_r}\right) , \tag{6}$$

where the last sum is over all possible sets of r replicas, counting once any permutation.

2.1.1 Interaction with a bimodal distribution.

We first study the following distribution for the J:

$$P(J) = \frac{1}{2}[\delta(J + J_0) + \delta(J - J_0)] \ . \tag{7}$$

Equation (6) is formally identical after averaging on the J, providing one sums only over the even r and writes J_0 instead of J.

If we define the overlaps

$$q_{a_1 \cdots a_r} = \frac{Tr_{\sigma_a} \sigma_{a_1} \cdots \sigma_{a_r} g_n^z(\{\sigma_a\})}{Tr_{\sigma_a} g_n^z(\{\sigma_a\})} \ , \tag{8}$$

we can write the eq. (3) as:

$$g_n(\{\sigma_a\}) = \cosh^n(\beta J) \sum_{r=0}^{n} \left(\tanh^r(\beta J) \sum_{a_1 < \cdots < a_r} \sigma_{a_1} \cdots \sigma_{a_r} q_{a_1 \cdots a_r}\right) \ . \tag{9}$$

We can now implement the $\frac{1}{z}$ expansion if we scale the couplings as usual:

$$J = \frac{\tilde{J}}{\sqrt{z}} \tag{10}$$

and set $\tilde{J}=1$. Performing the expansion, after some computations we obtain at the first order:

$$f = f_0 + \frac{1}{z} f_1 + O(\frac{1}{z^2}) , \tag{11}$$

with:

$$\beta f_0 = -\frac{\beta^2}{4} + \frac{\beta^2}{2n} \sum_{a<b} {q_{ab}^{(0)}}^2 - \frac{1}{n} \ln[Tr \exp(\beta \sum_{a<b} q_{ab}^{(0)} \sigma_a \sigma_b)] \tag{12}$$

and

$$\begin{aligned} \beta f_1 &= -\frac{\beta^2}{4} + \frac{\beta^4}{24} - \frac{\beta^2}{2n}(1 - \frac{5\beta^2}{3}) \sum_{a<b} {q_{ab}^{(0)}}^2 - \frac{\beta^4}{2n} \sum_{a<b<c<d} {q_{abcd}^{(0)}}^2 \\ &+ \frac{3\beta^4}{n} \sum_{a<b<c} (q_{ab}^{(0)} q_{bc}^{(0)} q_{ca}^{(0)}) + \frac{\beta^4}{n} \sum_{a<b<c<d} (q_{ab}^{(0)} q_{cd}^{(0)} + q_{ac}^{(0)} q_{bd}^{(0)} + q_{ad}^{(0)} q_{bc}^{(0)})\; q_{abcd}^{(0)} . \end{aligned} \tag{13}$$

As it should be (we are expanding around $z = +\infty$), f_0 is the SK free energy. In these expressions we have also expanded the overlaps in powers of $1/z$:

$$\begin{aligned} q_{ab} &= q_{ab}^{(0)} + \frac{1}{z} q_{ab}^{(1)} + \cdots , \\ q_{abcd} &= q_{abcd}^{(0)} + \frac{1}{z} q_{abcd}^{(1)} + \cdots \end{aligned} \tag{14}$$

and we have used the identities:

$$\begin{aligned} q_{ab}^{(0)} &= \frac{Tr_\sigma \exp[\beta^2 \sum_{r<s} q_{rs}^{(0)} \sigma_r \sigma_s] \sigma_a \sigma_b}{Tr_\sigma \exp[\beta^2 \sum_{r<s} q_{rs}^{(0)} \sigma_r \sigma_s]} \equiv < \sigma_a \sigma_b >_Q \qquad a \neq b \\ q_{abcd}^{(0)} &= \frac{Tr_\sigma \exp[\beta^2 \sum_{r<s} q_{rs}^{(0)} \sigma_r \sigma_s] \sigma_a \sigma_b \sigma_c \sigma_d}{Tr_\sigma \exp[\beta^2 \sum_{r<s} q_{rs}^{(0)} \sigma_r \sigma_s]} \equiv < \sigma_a \sigma_b \sigma_c \sigma_d >_Q \qquad a \neq b \neq c \neq d , \end{aligned} \tag{15}$$

where $< \cdot >_Q$ is the average on the single site Sherrington-Kirkpatrick Hamiltonian. We notice that f is no longer stationary with respect to the order parameter q because we have already used the stationary equations to simplify the result.

2.1.2 Interaction with a Gaussian distribution.

If we use a Gaussian distribution with the same mean ($\overline{J} = 0$) and variance (J_0^2/z) of the previously studied bimodal distribution, one finds that at this order the only relevant difference is in the fourth moment of the interaction ($3J_0^4/z^2$ for the Gaussian and J_0^4/z^2 for the bimodal one), (it is crucial that at this order we expand the order parameter g_n only up to the second order in z, i.e. up to the fourth order in J). Performing the same calculations as before, we arrive to the final form for the free energy first order correction:

$$\begin{aligned} \beta f_1 &= -\frac{\beta^2}{4} + \frac{\beta^4}{8} - \frac{\beta^2}{2n}(1 - 3\beta^2) \sum_{a<b} {q_{ab}^{(0)}}^2 - \frac{3\beta^4}{2n} \sum_{a<b<c<d} {q_{abcd}^{(0)}}^2 \\ &+ \frac{3\beta^4}{n} \sum_{a<b<c} (q_{ab}^{(0)} q_{bc}^{(0)} q_{ca}^{(0)}) + \frac{\beta^4}{n} \sum_{a<b<c<d} (q_{ab}^{(0)} q_{cd}^{(0)} + q_{ac}^{(0)} q_{bd}^{(0)} + q_{ad}^{(0)} q_{bc}^{(0)}) q_{abcd}^{(0)} . \end{aligned} \tag{16}$$

As expected, the f_0 does not change because it does not contain J^4 terms and the SK model is indeed independent from the particular distribution one uses if we fix the mean and the variance of the couplings.

2.2 Random lattice with fluctuating connectivity.

In an other interesting model the connectivity is a Poissonian variable with mean value z. We take into consideration the large z expansion, where the interactions probability distribution can be written in the form:

$$P(J_{ik}) = (1-\frac{z}{N})\delta(J_{ik}) + \frac{z}{N}\tilde{P}(J_{ik}) \qquad \forall i,k \quad , \tag{17}$$

where $\tilde{P}(J_{ik})$ is a distribution to be defined.

In principle we could write an expression similar to eq. (1) for the free energy, however it is simpler to proceed in a direct way. The n replicas partition function is:

$$\begin{aligned}\overline{Z^n} &= \prod_{i<k}\int_{-\infty}^{+\infty} P(J_{ik})dJ_{ik}Tr_\sigma \exp(\beta J_{ik}\sum_a \sigma_i^a\sigma_k^a) \\ &= \prod_{i<k} Tr_\sigma\left(1-\frac{z}{N}+\frac{z}{N}\int_{-\infty}^{+\infty} P(\tilde{J_{ik}})dJ_{ik}\exp(\beta J_{ik}\sum_a \sigma_i^a\sigma_k^a)\right) .\end{aligned} \tag{18}$$

2.2.1 The expression of the free energy

In the case where:

$$\tilde{P}(J_{ik}) = \frac{1}{2}[\delta(J_{ik}-J_0)+\delta(J_{ik}+J_0)] \qquad \forall i,k \tag{19}$$

we obtain:

$$\begin{aligned}\overline{Z^n} &= \prod_{i<k} Tr_\sigma\left(1+\frac{z}{N}[\cosh(\beta J_0\sum_a \sigma_i^a\sigma_k^a)-1]\right) = \\ &= Tr_\sigma \exp\{\frac{z}{N}\sum_{i<k}[\cosh(\beta J_0\sum_a \sigma_i^a\sigma_k^a)-1]\} .\end{aligned} \tag{20}$$

Let us rescale:

$$J_0 \to \frac{J_0}{\sqrt{z}} \tag{21}$$

and write $J_0 = 1$. We can than perform the $1/z$ expansion up to the first order for the free energy:

$$\begin{aligned}\overline{Z^n} &= Tr_\sigma \exp\{\frac{z}{N}\sum_{i<k}[\frac{\beta^2}{2z}(\sum_a \sigma_i^a\sigma_k^a)^2 + \frac{\beta^4}{24z^2}(\sum_a \sigma_i^a\sigma_k^a)^4]\} = \\ &= Tr_\sigma \exp\{\frac{1}{N}\sum_{i<k}[\frac{\beta^2}{2}(\sum_{a,b} \sigma_i^a\sigma_k^a\sigma_i^b\sigma_k^b) + \frac{\beta^4}{24z}(\sum_{a,b,c,d} \sigma_i^a\sigma_k^a\sigma_i^b\sigma_k^b\sigma_i^c\sigma_k^c\sigma_i^d\sigma_k^d)]\} .\end{aligned} \tag{22}$$

After converting the summations over replicas indices into distinct indices summations (using $(\sigma^a)^2 = 1$), introducing the Gaussian integrals and solving with the saddle point method, we find at the first order: $\beta f = \beta f_0 + \frac{1}{z}\beta f_1$, where f_0 is the SK free energy and f_1 has the form:

$$\beta f_1 = \beta f_1^J \equiv \frac{\beta^4}{24} + \frac{\beta^4}{3n}\sum_{a<b} {q_{ab}^{(0)}}^2 - \frac{\beta^4}{2n}\sum_{a<b<c<d} {q_{abcd}^{(0)}}^2 \ . \tag{23}$$

In a similar way, when $P(J)$ is a Gaussian distribution, we find the same result as before with the difference that the first order free energy f_1 is multiplied by a factor three.

If we put together the previous formulae we find that

$$\beta f_1 = A f_1^{neigh} + K f_1^J \tag{24}$$

where $A = 1$ for the model with fixed number of neighbours, $A = 0$ for the model with fluctuating number of neighbours and K is the kurtosis of the distribution of couplings J. The quantity f_1^J is given by eq. (23) while f_1^{neigh} by the formulae 24 and 16 is found to be:

$$\begin{aligned} \beta f_1^{neigh} &= -\frac{\beta^2}{4} - \frac{\beta^2}{2n}(1-\beta^2)\sum_{a<b} {q_{ab}^{(0)}}^2 \\ &+ \frac{3\beta^4}{n}\sum_{a<b<c}(q_{ab}^{(0)}q_{bc}^{(0)}q_{ca}^{(0)}) + \frac{\beta^4}{n}\sum_{a<b<c<d}(q_{ab}^{(0)}q_{cd}^{(0)} + q_{ac}^{(0)}q_{bd}^{(0)} + q_{ad}^{(0)}q_{bc}^{(0)})q_{abcd}^{(0)} \ . \end{aligned} \tag{25}$$

3 Evaluation of the sums over replica's indices.

If the replica is broken at two steps (using the usual conventions) we can write:

$$\lim_{n\to 0}\frac{1}{n}\sum_{ab} q_{ab}^2 = (m_2-1)q_2^2 + (m_1-m_2)q_1^2 + m_1 q_0^2 \ , \tag{26}$$

where the sum is on the indices $a \neq b$. Similar expressions can be written at higher orders in the replica symmetry breaking and in the continuum limit one obtains:

$$\lim_{n\to 0}\frac{1}{n}\sum_{ab} q_{ab}^2 = \int_0^1 q^2(x). \tag{27}$$

The formulae for the case where the replica symmetry is broken at two steps can be obtained by using:

$$q(x) = q_0 \qquad 0 \le x < m_1 \tag{28}$$

$$q(x) = q_1 \qquad m_1 \le x < m_2 \tag{29}$$

$$q(x) = q_2 \qquad m_2 \le x \le 1 \tag{30}$$

The direct computation of

$$\lim_{n\to 0}\frac{1}{n(n-1)(n-2)(n-3)}\sum_{abcd} q_{abcd}^2 \tag{31}$$

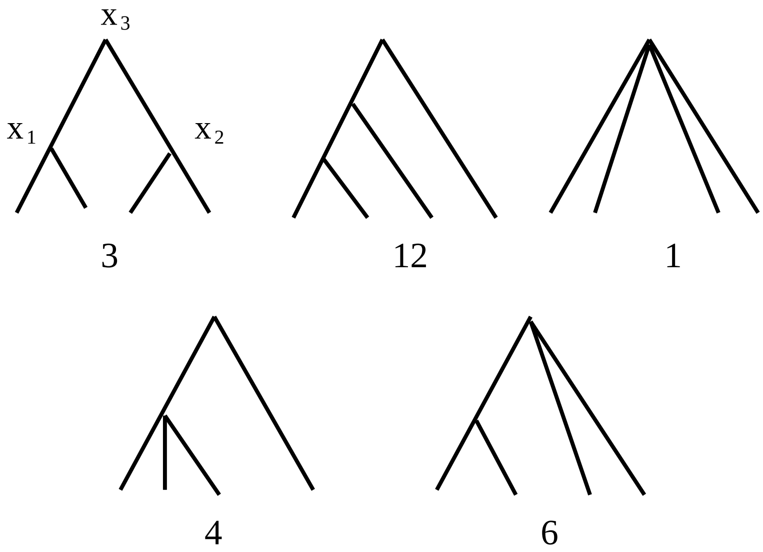

Figure 1: Four replicas diagrams. The number below the trees is the degeneration due to the replicas indices permutations.

is more involved and if it is not properly done it can become a nightmare.

We can simplify it if we remark that the four replicas overlap is a function of all the possible x's among the four replicas:

$$q_{abcd} = q(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}), \tag{32}$$

with $0 \leq x_{ij} \leq 1 \ \forall i, j$; however, due to the ultrametric structure of the states at most three of the two replicas overlaps can be distinct. Mézard and Yedidia [10] has shown that in order to compute this kind of sums over replicas is suitable to consider the five possible ways in which four replicas can be organized (fig. 1); we have to associate a variable x_i to each vertex and a factor $x_i^{s-2}(s-2)!$ when s lines converge to it. Using this rule, providing to take into account the number of different permutations of replicas indices that produce the same configuration, we finally obtain:

$$\begin{aligned}
&\lim_{n\to 0} \frac{1}{n(n-1)(n-2)(n-3)} \sum_{abcd} q^2_{abcd} = 3\int_1^0 dx_3 \int_1^{x_3} dx_2 \int_1^{x_3} dx_1 q(x_1, x_2, x_3)^2 + \\
+ \quad & 12\int_1^0 dx_3 \int_1^{x_3} dx_2 \int_1^{x_2} dx_1 q(x_1, x_2, x_3)^2 + \int_1^0 dx_1 2x_1^2 q(x_1^2) + \\
+ \quad & 4\int_1^0 dx_2 \int_1^{x_2} x_1 dx_1 q(x_1, x_2)^2 + 6\int_1^0 x_2 dx_2 \int_1^{x_2} dx_1 q(x_1, x_2)^2 \quad .
\end{aligned} \tag{33}$$

At the second step of replica symmetry breaking we obtain:

$$\begin{aligned}
&\tfrac{24}{n} \quad \sum_{a<b<c<d} q^2_{abcd} = (m_2-1)(m_2-2)(m_2-3)q^2_{4_2} \\
+ \quad & 3(1-m_2)^2(m_1-m_2)q^2_{2_2 2_2 sb} - 3m_1(1-m_2)^2 q^2_{2_2 2_2 bd}
\end{aligned}$$

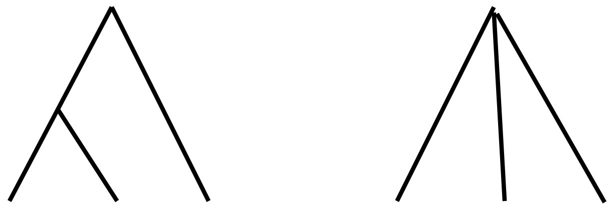

Figure 2: Three replicas diagrams.

$$
\begin{aligned}
- &\ 6(1-m_2)(m_2-m_1)(2m_2-m_1)q^2_{2_2 2_1 sb} - 6(1-m_2)(m_2-m_1)m_1 q^2_{2_2 2_1 bd} \\
- &\ 12(1-m_2)m_1^2 q^2_{2_2 2_0} + (m_1-m_2)(m_1-2m_2)(m_1-3m_2)q^2_{4_1} \\
- &\ 3(m_2-m_1)^2 m_1 q^2_{2_1 2_1} - 12(m_2-m_1)m_1^2 q^2_{2_1 2_0} - 6m_1^3 q^2_{4_0} \\
+ &\ 4(m_2-1)(m_2-2)(m_1-m_2)q^2_{3_2 1_1} - 4m_1(m_2-1)(m_2-2)q^2_{3_2 1_0} \\
- &\ 4m_1(m_1-m_2)(m_1-2m_2)q^2_{3_1 1_0} - 12m_1(1-m_2)(m_2-m_1)q^2_{2_2 1_1 1_0}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\tfrac{24}{n} &\sum_{a<b<c<d} q_{abcd}(q_{ab}q_{cd} + q_{ac}q_{bd} + q_{ad}q_{bc}) = \\
&\ 3(m_2-1)(m_2-2)(m_2-3)q_{4_2}q_2^2 + 3(1-m_2)^2(m_1-m_2)q_{2_2 2_2 sb}(q_2^2+2q_1^2) \\
- &\ 4m_1(1-m_2)^2 q_{2_2 2_2 bd}(q_2^2+2q_0^2) - 6(1-m_2)(m_2-m_1)(2m_2-m_1)q_{2_2 2_1 sb}(q_2q_1+2q_1^2) \\
- &\ 6(1-m_2)(m_2-m_1)m_1 q_{2_2 2_1 bd}(q_2q_1+2q_0^2) - 12(1-m_2)m_1^2 q_{2_2 2_0}(q_2q_0+2q_0^2) \\
+ &\ 3(m_1-m_2)(m_1-2m_2)(m_1-3m_2)q_{4_1}q_1^2 - 3(m_2-m_1)^2 m_1 q_{2_1 2_1}(q_1^2+2q_0^2) \\
- &\ 12(m_2-m_1)m_1^2 q_{2_1 2_0}(q_1q_0+2q_0^2) - 18m_1^3 q_{4_0}q_0^2 \\
+ &\ 12(m_2-1)(m_2-2)(m_1-m_2)q_{3_2 1_1}q_2q_1 - 12m_1(m_2-1)(m_2-2)q_{3_2 1_0}q_2q_0 \\
- &\ 12m_1(m_1-m_2)(m_1-2m_2)q_{3_1 1_0}q_1q_0 \\
- &\ 12m_1(1-m_2)(m_2-m_1)q_{2_2 1_1 1_0}(q_2q_0+2q_1q_0)\ ,
\end{aligned} \tag{35}
$$

where in the notation $q_{A_a\cdots}$ the quantity A is the number of replicas and a indicates the block to which they belong (we refer to the matrix Q_{ab} at the second step of the ultrametric Ansatz); when two possibilities can occur, we write sb when the four replicas are in the same block of first replica symmetry breaking and bd in the other case (i.e. $q_{2_2 2_1 sb}$ means that two replicas belong to the same second replica symmetry breaking block and all the four to the same block of first replica symmetry breaking; whereas $q_{2_2 2_1 bd}$ means that two replicas belong again to the same second replica symmetry breaking block, the other two to the same first replica symmetry breaking block but the overlap between the first two and the second two is the minimum one).

For the sum on three replicas indices we have (see fig.(2)):

$$
\begin{aligned}
\tfrac{6}{n} &\sum_{a<b<c} (q^{(0)}_{ab} q^{(0)}_{bc} q^{(0)}_{ca}) = (m_2-1)(m_2-2)q_2^3 \\
+ &\ 3(1-m_2)(m_2-m_1)q_2q_1^2 + 3(1-m_2)m_1q_2q_0^2 \\
+ &\ (m_1-m_2)(m_1-2m_2)q_1^3 + 3m_1(m_2-m_1)q_1q_0^2 + 2m_1^2q_0^3\ .
\end{aligned} \tag{36}
$$

Substituting these expressions into 13, 16, 23, we obtain the explicit expression for the free energies. We can now find the numerical values of q_2, q_1, q_0, m_2, m_1 maximizing f_0 (the SK Hamiltonian is stationary with respect to $q(x)$) and then we can use these values in the expressions of the four replicas overlaps.

To obtain the expressions at one level of RSB we can put $q_2 = q_1$ and identify the four replicas overlaps in this way: $q_{4_2} = q_{4_1} = q_{2_2 2_2 sb} = q_{2_2 2_1 sb} = q_{3_2 1_1}$; $q_{3_2 1_0} = q_{3_1 1_0} = q_{2_2 1_1 1_0}$; $q_{2_2 2_2 bd} = q_{2_1 2_1} = q_{2_2 2_1 bd}$; $q_{2_2 2_0} = q_{2_1 2_0}$ (see [7]).

4 The solution of the equation

To evaluate the value of the free energy we have firstly to solve for the m and q parameters of the infinite connectivity limit and to compute the parameters in equations (27), (34), (35), (36). At this end we have to compute integrals like the following:

$$
\begin{aligned}
q_{2_2 2_2 sb} &= \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi q_0}} \exp(-\frac{z^2}{2q_0}) \left[\frac{\mathrm{Num}}{\mathrm{Den}}\right] \\
&\quad \text{where} \\
\mathrm{Num} &= \int_{-\infty}^{+\infty} dy \exp(-\frac{y^2}{2(q_1-q_0)}) \left\{ \int_{-\infty}^{+\infty} dx \exp(-\frac{x^2}{2(q_2-q_1)}) \cosh^{m_2}(\beta(z+y+x)) \right\}^{\frac{m_1}{m_2}-2} \\
&\times \left\{ \int_{-\infty}^{+\infty} dx \exp(-\frac{x^2}{2(q_2-q_1)}) \tanh^2(\beta(z+y+x)) \cosh^{m_2}(\beta(z+y+x)) \right\}^2 \\
&\quad \text{and} \\
\mathrm{Den} &= \int_{-\infty}^{+\infty} dy \exp(-\frac{y^2}{2(q_1-q_0)}) \left\{ \int_{-\infty}^{+\infty} dx \exp(-\frac{x^2}{2(q_2-q_1)}) \cosh^{m_2}(\beta(z+y+x)) \right\}^{\frac{m_1}{m_2}}
\end{aligned}
\tag{37}
$$

To evaluate these expressions it is important to optimize the number of operations the computer has to do. In the numerical evaluation of the integrals (we are considering sums instead of integrals and we set $x = a\ i$ where i is an integer) the internal integrals have in fact to be evaluated for every value of the variable of the external one. A repeated evaluation would take an enormous amount of time. A much faster method consists in evaluating beforehand the internal functions (i.e. $\cosh(x+y+z)$) for all values $x+y+z = a\ i$ and in storing in a table the values in the integrals; in this way the computer has to perform a number of operations proportional to N instead to N^3 of the naive method (N is the number of spacings in which the integration domain is divided).

The final results for the coefficient of the $1/z$ corrections to the free energy are shown in fig 3 in the case of ± 1 interactions. We immediately see that the divergence of the correction to the free energy at $T = 0$ fades away when we increase the order of the replica breaking and it is an artefact of using a starting point which is not correct (the correct one corresponds to infinite breaking of the replica symmetry).

We evaluated the entropy doing the derivative of the free energy ($S = -df/dT$) using an high order expression for the finite difference derivative. The final results for the coefficient

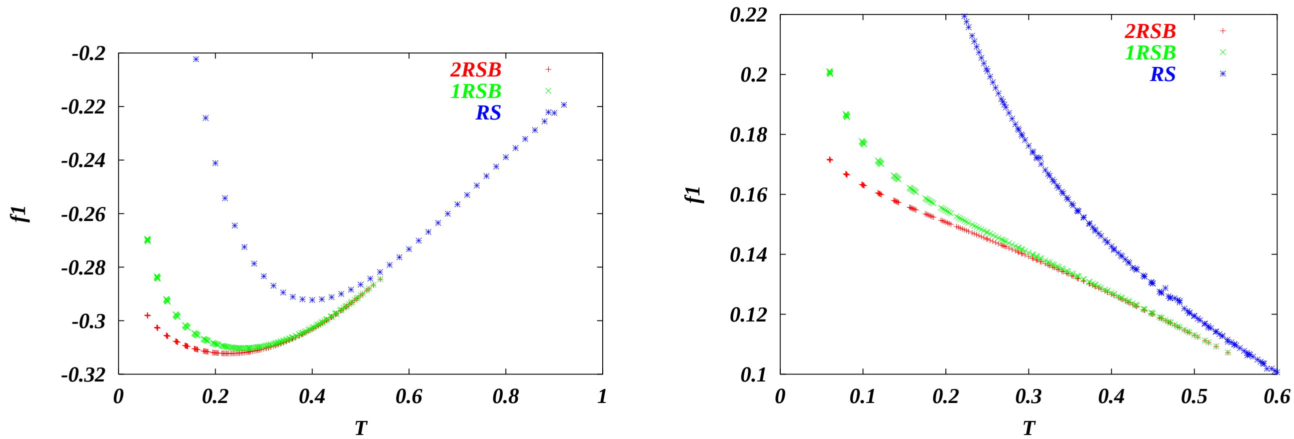

Figure 3: The $1/z$ correction to the free energy as function of the temperature for the replica symmetric case $(*)$, one step replica symmetry breaking and two steps replica symmetry breaking for the model with $J = \pm 1$ for fixed connectivity (left) and for fluctuating connectivity (right),

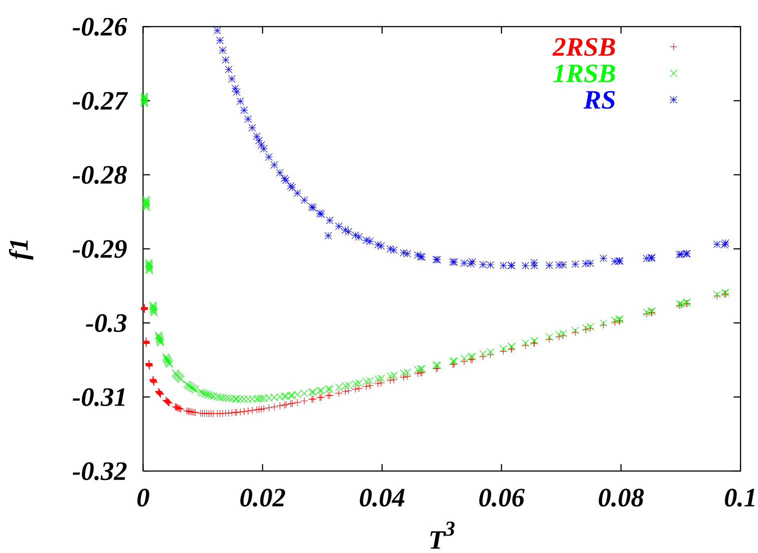

Figure 4: The $1/z$ correction to the free energy as function of T^3 for the replica symmetric case $(*)$, one step replica symmetry breaking and two steps replica symmetry breaking for the model with $J = \pm 1$ for fixed connectivity

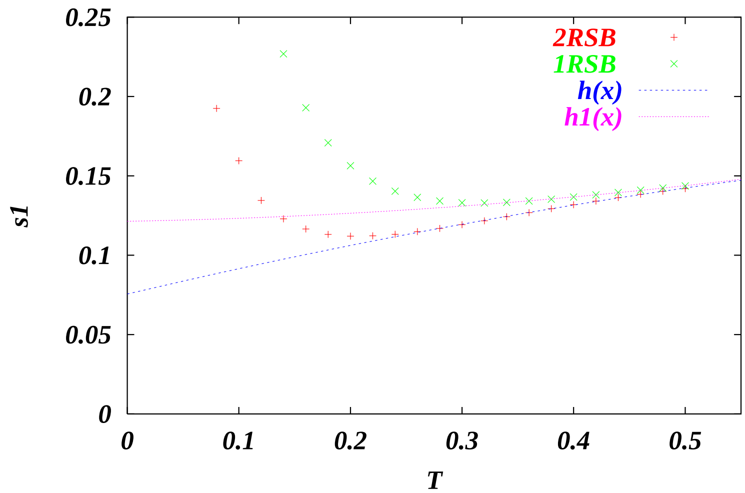

Figure 5: The $1/z$ correction to the entropy as function of the temperature for one step replica symmetry breaking and two steps replica symmetry breaking for the model with $J = \pm 1$ for fixed connectivity and the corresponding second order polynomial fits in the high temperature region.

of the $1/z$ corrections to the free energy are shown in fig 5 in the case of ± 1 interactions with fluctuating connectivity. Also in this case we see that the divergence of the correction to the entropy near $T = 0$ fades away when we increase the order of the replica breaking. The correction for the entropy are much stronger that those for the free energy. In order to evidentiate the effect of the spurious divergence at $T = 0$ we show also a second order polynomial fit in the high temperature region, which dramatically fails at low temperature.

5 Conclusions.

We can see from the numerical data that the first order correction f_1 of the free energy in all the analyzed models taken into account presents a divergence at small temperatures. We fit in the interval $T \in [0.05, 0.5]$ a behaviour of the kind (see fig.(3)):

$$f_1(T) = D + AT^3 + BT^2 + \frac{C}{T} \, . \tag{38}$$

We report in the following table the values of the C coefficient in the different cases considered.

C values				
	fixed connectivity		fluct. connectivity	
	± 1	Gaussian	± 1	Gaussian
RS	0.035	0.12	0.35	0.12
1RSB	0.003	0.010	0.003	0.010
2RSB	0.001	0.003	0.001	0.003

This coefficient is three times smaller when going from the 1RSB solution to the 2RSB one. Moreover, if we look at the results in [5] we can see that the same ratios have been found for the $1/\sqrt{z}$ coefficient; its values are in fact 0.1 for RS, 0.01 for 1RSB and 0.0026 for 2RSB solutions. The same divergent factors occur in the fixed and in the fluctuating connectivity models and the entire divergence comes from the term proportional the forth moment of the distribution of the J. However, we can see from the figures that the divergence moves to smaller temperatures when the number of replica symmetry breaking steps increases. In the fixed connectivity model with bimodal distribution the divergence appears at $T < 0.1$.

We notice that in all these models there is a T^2 correction to the low temperature behaviour of the S.K. model, where the free energy is proportional to T^3. From the numerical data suggest that this correction goes to zero in the full replica symmetry breaking solution, at least in the fixed connectivity model, that is linear in T^3 over a large range of temperature already in the 2RSB solution (fig.(4)).

We fitted the curves in the temperature interval where the divergence doesn't yet occur, with the function:

$$f_1(T) = D + AT^3 + BT^2 \ . \tag{39}$$

For completeness we report in the following table the values of the coefficients A and B in the different cases considered.

B values				
	fixed connectivity		fluct. connectivity	
	± 1	Gaussian	± 1	Gaussian
RS	-0.199	-0.049	-0.63	-1.88
1RSB	-0.122	-0.139	-0.37	-1.12
2RSB	-0.054	-0.08	-0.3	-0.9

A values				
	fixed connectivity		fluct. connectivity	
	± 1	Gaussian	± 1	Gaussian
RS	-0.286	-0.156	0.19	0.619
1RSB	-0.30	-0.176	0.165	0.497
2RSB	-0.313	-0.18	0.16	0.48

The fit we have done assumes that, apart form the $1/T^2$ divergence, the entropy extrapolates to zero at zero temperature[1]. In order to check the consistency of the results, we have extrapolated to zero temperature the numerical results for the entropy in the temperature interval where the divergence doesn't appear yet. We then find a behaviour that accords with the expected one: the zero temperature value $s(0)$ is different from zero but decreases when going to higher steps of replica symmetry breaking, suggesting that it will reach the correct value $s_1(0) = 0$ in the infinite steps limit.

We give the results in the following table.

1st order correction to entropy at $T = 0$				
	fixed connectivity		fluct. connectivity	
	± 1	Gaussian	± 1	Gaussian
1RSB	0.26	0.48	0.12	0.36
2RSB	0.18	0.32	0.07	0.22

To conclude, we think that there are numerical evidences that confirm that the $1/z$ expansion is correct to study the random lattices with high connectivity. The $1/z$ expansion arise naturally if we compare the high connectivity limit in random lattices to the case of nearest neighbor interactions in the high dimension limit. The high dimension expansion is indeed in powers of $1/D$ and it coincides at the first order in D with the $1/z$ expansion.

6 Appendix.

In order to demonstrate that (1) is the correct functional for the free energy we can show that $\lim_{n\to 0} \frac{1}{n}\partial(\beta f)/\partial\beta$ is the internal energy when g_n is solution of (3) and that (1) is correctly normalized at $\beta = 0$ ($-\beta f(\beta = 0) = \ln 2$); the latter condition is easy to verify considering that $g_n(\beta = 0) = 1$ and that Tr_{σ_a} gives 2^n terms. To convince oneself of the validity of the former assertion one can construct explicitly the order parameter $g_n(\{\sigma_0^a\})$ making clear its physical meaning. Following the approach of [9] we start writing the partition function in a recursive manner making use of the equivalence of the model with a Cayley tree. Focusing on an arbitrary spin σ_0:

$$Z = \sum_{\{\sigma\}} \exp\left(\beta h \sigma_0\right) \prod_{k=1}^{z+1} Q_{(L)}(\sigma_0|\sigma^{(k)}) \ , \tag{40}$$

where

$$Q_{(L)}(\sigma_0|\sigma^{(1)}) = \exp\left(\beta J_{01}\sigma_0\sigma_1 + \beta h \sigma_1\right) \prod_{k=1}^{z} Q_{(L-1)}(\sigma_1|\sigma^{(k)}) \ . \tag{41}$$

[1]This is true for the Gaussian model, but it is not true for the ± 1 at fixed z, where *spin fou* are present. However it is reasonable that this difference can be seen only at higher orders in the $1/z$ expansion.

$z+1$ is the branches number (random lattice's connectivity) and L the shells number; the $\sigma^{(k)}$ are the spins on k-th branch excluding the σ_0; h is an external uniform field.

We can than write the n replicas partition function:

$$Z^n = \prod_{a=1}^{n} Z_a = \sum_{\{\sigma_1\}} \cdots \sum_{\{\sigma_n\}} \exp\left(\sum_{a=1}^{n} \beta h \sigma_0^a\right) \prod_{k=1}^{z+1} \prod_{a=1}^{n} Q_{(L)}(\sigma_0^a | {\sigma^{(k)}}^a) \tag{42}$$

and define:

$$g_{n,(L)}(\{\sigma_0^a\}) \equiv \overline{\sum_{\{{\sigma^{(k)}}^a\}} \prod_{a=1}^{n} Q_{(L)}(\sigma_0^a | {\sigma^{(k)}}^a)} \quad , \tag{43}$$

where the bar is the average over the random couplings J.

From (42) and (43) we obtain:

$$\overline{Z^n} = \sum_{\{\sigma_0^a\}} \exp\left(\sum_{a=1}^{n} \beta h \sigma_0^a\right) g_{n,(L)}^{z+1}(\{\sigma_0^a\}) \quad , \tag{44}$$

that reveals $g_{n,(L)}(\{\sigma_0^a\})$ to be the one branch contribution to the partition function.

By definition it follows the recursion relation:

$$g_{n,(L)}(\{\sigma_0^a\}) = \int_{-\infty}^{+\infty} dJ P(J) \sum_{\{\sigma_1^a\}} \exp\left(\sum_{a=1}^{n} \beta h \sigma_1^a + \sum_{a=1}^{n} \beta J \sigma_0^a \sigma_1^a\right) g_{n,(L-1)}^{z}(\{\sigma_1^a\}) \ . \tag{45}$$

The internal energy density can be written as a bond energy multiplied by the number of links per spin ($(z+1)/2$).

If we consider a link with a coupling constant J between two spins σ_0 and σ_1, we can write its energy as [9]:

$$E_{01} = -J < \sigma_0 \sigma_1 > . \tag{46}$$

The expectation value is computed with the Hamiltonian $H = -J\sigma_0\sigma_1 + H_0 + H_1$, where H_0 is the Hamiltonian of the spin σ_0 before being connected with σ_1 and can be written as $H_0 = -\ln\left(g_n(\{\sigma_0\})\right)^z/\beta$; the same argument can be repeated for σ_1.

At this level we should use the finite normalized order parameter:

$$g_{n,(L)}(\{\sigma_0^a\}) \equiv \frac{\overline{\sum_{\{{\sigma^{(k)}}^a\}} \prod_{a=1}^{n} Q_{(L)}(\sigma_0^a | {\sigma^{(k)}}^a)}}{\overline{\sum_{\{{\sigma_k}^a\}} \prod_{l=1}^{z} \sum_{\{{\sigma^{(l)}}^a\}} \prod_{a=1}^{n} Q_{(L-1)}(\sigma_k^a | {\sigma^{(l)}}^a)}} \tag{47}$$

which follows the recursion equation:

$$g_{n,(L)}(\{\sigma_0^a\}) = \frac{\int_{-\infty}^{+\infty} dJ P(J) \sum_{\{\sigma_1^a\}} \exp\left(\sum_{a=1}^{n} \beta J \sigma_0^a \sigma_1^a\right) g_{n,(L-1)}^{z}(\{\sigma_1^a\})}{\sum_{\{\sigma_1^a\}} g_{n,(L-1)}^{z}(\{\sigma_1^a\})} \ . \tag{48}$$

In the thermodynamic limit, taking into account only the inner part of the Cayley tree, the g_n are shells independent, so we can write:

$$g_n(\{\sigma_a\}) = \frac{\int_{-\infty}^{+\infty} dJ P(J) \sum_{\{\tau_a\}} \exp\left(\sum_{a=1}^{n} \beta J \sigma_a \tau_a\right) g_n^z(\{\tau_a\})}{\sum_{\{\tau_a\}} g_n^z(\{\tau_a\})} \ . \tag{49}$$

Anyway the internal energy (as the free energy) is insensitive to the normalization of the order parameter, so we can use both (49) or (45) (in the last one the thermodynamic limit has to be taken).

We can now evaluate the derivative of βf with respect to β and appurate that we obtain the expression we expected:

$$n\frac{\partial(\beta f)}{\partial \beta} = \frac{z+1}{2} \times \tag{50}$$
$$\times \frac{\int_{-\infty}^{+\infty} dJ P(J) Tr_{\{\sigma_a\}} Tr_{\{\tau_a\}} g_n^z(\{\sigma_a\}) g_n^z(\{\tau_a\}) \exp\left[\beta J \sum_{a=1}^n \sigma_a \tau_a\right](-J\sum_{a=1}^n \sigma_a \tau_a)}{\int_{-\infty}^{+\infty} dJ P(J) Tr_{\{\sigma_a\}} Tr_{\{\tau_a\}} g_n^z(\{\sigma_a\}) g_n^z(\{\tau_a\}) \exp\left[\beta J \sum_{a=1}^n \sigma_a \tau_a\right]}.$$

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